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Algebraic structures behind Hilbert schemes and wreath products

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Abstract. In this paper we review various strikingly parallel algebraic structures behind Hilbert schemes of points on surfaces and certain finite groups called the wreath products. We explain connections among Hilbert schemes, wreath products, infinite-dimensional Lie algebras, and vertex algebras. As an application we further describe the cohomology ring structure of the Hilbert schemes. We organize this paper around several general principles which have been supported by various works on these subjects.

Contents

1. Introduction

The purpose of this paper is to describe the analog and connections between two seemingly unrelated subjects. The first one concerns about a wonderful geometric object, namely, the Hilbert scheme $X^{[n]}$ of n points on a (quasi-)projective surface X. It has been well known [**Fog**] that $X^{[n]}$ is non-singular of complex dimension $2n$. The Hilbert-Chow morphism from $X^{[n]}$ to the symmetric product X^n/S_n is a semismall crepant resolution of singularities. The second one is a finite group called the wreath product Γ_n, namely, the semi-direct product between the symmetric group S_n and the product group Γ^n of a finite group Γ. The representation theory of wreath products was first developed by Specht [**Spe**], also cf. e.g. [**Mac, Zel**].

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For Γ trivial, Γ_n reduces to S_n. Many nice features of the representation theory of symmetric groups remain to hold in the setup of wreath products.

It turns out that there exist deep connections among the geometry of Hilbert schemes, representation theory of wreath products, and vertex algebras. In recent years, these subjects have attracted various people with diversified backgrounds such as algebraic geometry, representation theory, combinatorics, and mathematical physics. In this mostly expository paper, we attempt to organize our discussions around four main points which have been supported by various works on these subjects. (It also includes some original materials from the unpublished paper [**Wa2**] and further clarification). It is our hope that many topics under discussion may attain somewhat better understanding in this way and these principles may serve again as a helpful guide in uncovering new structures.

The *first point* is that one should study the Hilbert schemes $X^{[n]}$ (respectively the wreath products Γ_n) for all $n \geq 0$ simultaneously. It was not too long ago when people started to study the geometry of Hilbert schemes in such a way. Göttsche [**Got1, Got2**] calculated the Betti numbers for the Hilbert scheme $X^{[n]}$ associated to a surface X and presented it in a beautiful formula in terms of a generating function for all $n \geq 0$ together. Motivated by Göttsche's formula, quiver varieties [**Na1**] and Vafa-Witten's paper on S-duality [**VW**], Nakajima [**Na2, Na3**] constructed a Heisenberg algebra associated to the lattice $H^*(X, \mathbb{Z})/tor$ in terms of the correspondence varieties which act irreducibly on the direct sum $\mathbb{H}_X$ of the cohomology groups of $X^{[n]}$ for all $n \geq 0$. Similar results were obtained by Grojnowski [**Gro**]. On the other hand, it has been known for a long time that one should study the representations of symmetric groups S_n, or more generally the wreath products Γ_n, for all $n \geq 0$ together, cf. [**Spe, Zel, Mac**]. A Heisenberg algebra associated to the lattice $R_{\mathbb{Z}}(\Gamma)$ was constructed by the author [**Wa1**] (also cf. [**FJW1**]) acting irreducibly on the direct sum $\mathbb{R}_\Gamma$ of representation rings of Γ_n for all $n \geq 0$. Here $R_{\mathbb{Z}}(\Gamma)$ denotes the integral span of the irreducible characters of Γ.

The *second point* is that the geometry/invariants of a suitable resolution of singularities of an orbifold should be compared to the equivariant counterparts of the orbifold. This was largely stimulated by the study of the orbifold string theory where the notion of orbifold Euler numbers was introduced [**DHVW**], and it may also be viewed as a McKay correspondence in a broad sense, cf. [**Rei1, Rei2**] and references therein. Given an orbifold, one may study the associated equivariant K-group, equivariant derived category, orbifold cohomology, orbifold Euler number, orbifold Hodge number, orbifold elliptic genera etc, and compare them with their counterparts on a suitable resolution, cf. [**DHVW, Zas, DMVV, Bat, BKR, Wa2, CR**] and references therein. The Euler numbers and Hodge numbers of Hilbert schemes have been computed in [**Got1, GS**]. The Hilbert-Chow morphism $X^{[n]} \to X^n/S_n$ provides a great example which matches invariants on a suitable resolution with those on the orbifold, cf. [**HH, Got3, DMVV, Zhou**].

The analog and connections between Hilbert schemes and wreath products were pointed out by the author [**Wa1**]. Given a space Y with an action by a finite group Γ, the wreath product Γ_n acts on the n-th direct product Y^n in a canonical way. Below we assume $\tau : X \to Y/\Gamma$ is a resolution of singularities of the orbifold Y/Γ and assume both X and Y are complex surfaces. We observe that there is a naturally induced resolution of singularities $\tau_n : X^{[n]} \to Y^n/\Gamma_n$. The *third point* is that whenever the resolution $\tau : X \to Y/\Gamma$ is 'good' in a suitable sense then

the resolution $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ is 'good'. A variant of this can be formulated as follows: whenever a certain 'reasonable' statement can be made relating X and Y/Γ then the corresponding statement relating $X^{[n]}$ and Y^n/Γ_n should hold.

For example, if a resolution $\tau : X \to Y/\Gamma$ is crepant, respectively semismall, then so is the corresponding resolution $\tau_n : X^{[n]} \to Y^n/\Gamma_n$. If the Euler number of X is equal to the orbifold Euler number of the orbifold Y/Γ, then the Euler number of $X^{[n]}$ is equal to the orbifold Euler number of the orbifold Y^n/Γ_n, cf. [**Wa1**]. The statement remains to be true by considering Hodge numbers instead of Euler numbers, and it is conjectured the statement is also true for the elliptic genera, cf. Wang-Zhou [**WaZ**]. We remark that Borisov and Libgober [**BL**] have formulated mathematically the notion of an orbifold elliptic genera and verified a conjectural formula in [**WaZ**] for the orbifold elliptic genera of Y^n/Γ_n. When Γ is trivial and X equals Y, the third principle above simply says the Hilbert-Chow resolution is 'good' in all aspects mentioned in the previous paragraphs. Since one can easily construct various examples of good resolutions $\tau : X \to Y/\Gamma$, this principle provides in a tautological way numerous new higher dimensional examples of good resolutions. To our best knowledge, (except a few isolated cases) all the known higher dimensional examples of good resolutions arise in such a way.

This above discussion 'explains' why it is natural to study the direct sum of the equivariant K-groups $K_{\Gamma_n}(Y^n) \bigotimes \mathbb{C}$ and why this should be parallel to the study of $\mathbb{H}_X$ [**Wa1**] (also cf. [**Seg2**] and a footnote in [**Gro**] for the important case when Γ is trivial). To simplify the discussion, we will refrain ourselves to discuss the representation rings $R(\Gamma_n)$ (instead of the equivariant K groups) with connections to the cohomology groups of Hilbert schemes. In an important case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, Y is the affine plane $\mathbb{C}^2$, and X is the minimal resolution of singularities of the simple singularity $\mathbb{C}^2/\Gamma$, considering only the representation rings $R(\Gamma_n)$ does not really lose information since the equivariant K group $K_{\Gamma_n}(\mathbb{C}^{2n}) \bigotimes \mathbb{C}$ is isomorphic to $R(\Gamma_n)$ by the Thom isomorphism. This lead the author to propose a group theoretic realization, which can be viewed as a new variation of McKay correspondence and have been subsequently developed jointly with I. Frenkel and Jing [**FJW1**], of the homogeneous vertex representation of the affine Kac-Moody algebra of ADE types and its toroidal counterpart (cf. [**FK, Seg1, MRY**]). We remark that a weighted bilinear form on $R(\Gamma_n)$ introduced in [**FJW1**] in a group theoretic manner affords a natural interpretation in terms of the Koszul-Thom complex [**Wa2**]. This approach to the McKay correspondence is parallel to a geometric realization in terms of Hilbert schemes $X^{[n]}$ when X is the minimal resolution $\mathbb{C}^2//\Gamma$ of the simple singularity $\mathbb{C}^2/\Gamma$, cf. [**Na3**].

The *fourth point* is that both the Hilbert schemes and wreath products have deep connections with the theory of vertex algebras [**Bor1**][1], and it is beneficial to study them in a parallel way. The shift from a representation to its underlying symmetry algebra provides new insights into various structures of the representation. In the framework of Hilbert schemes, the first indication of connections with vertex algebras comes from the construction of the Heisenberg algebra (cf. [**Na2, Gro, Na3**]). The Heisenberg algebra construction was used by de Cataldo

[1]Vertex algebras have played an important role in many different fields, including string theory, infinite-dimensional Lie algebras, the Monster group, the moonshine conjecture, mirror symmetry, moduli space of algebraic curves, and others, cf. [**Kac1, FLM, Bor2, Kac2, Fre**] and the references therein.

and Migliorini [**dCM**] to study the geometry of Hilbert schemes from a novel viewpoint. Lehn [**Lehn**] realized geometrically the Virasoro algebra and applied it to study connections between Heisenberg generators and cup products on the cohomology groups of Hilbert schemes. In our joint work with Li and Qin [**LQW1, LQW2**], we have further developed the connection between vertex algebras and Hilbert schemes and used it to obtain new results on the cohomology ring structure of $X^{[n]}$ associated to an arbitrary projective surface X (which in general has been unaccessible by classical methods in algebraic geometry). On the other hand, in the framework of wreath products, there has been a group theoretic construction of the Virasoro algebra given by I. Frenkel and the author [**FW**] acting on $\mathbb{R}_\Gamma$ which uses the construction of Heisenberg algebra in [**Wa1**]. The representation theory of symmetric groups was earlier studied from the viewpoint of vertex operators by I. Frenkel and Jing (cf. [**Jing**]), and the natural question of understanding further connections between vertex algebras and symmetric groups has been posted since then. Although there has been much nontrivial evidence indicating deep connections among $\mathbb{H}_X$, $\mathbb{R}_\Gamma$, and vertex algebras, much remains to be done to go beyond the vertex representation level in both pictures[2].

The wreath products are relatively simple objects, while the geometry of Hilbert schemes is very rich. The strikingly parallel algebraic structures appearing in both $\mathbb{H}_X$ and $\mathbb{R}_\Gamma$ have been and will still be a mutual stimulation to study these subjects together. For instance, the group-theoretic construction of the Virasoro algebra in [**FW**] was motivated in part by Lehn's geometric construction in [**Lehn**]. The work [**FW**] in addition suggested the existence of a surprising relation between the cup product on the cohomology ring of Hilbert schemes and the convolution product on the representation ring of wreath products. A precise connection between the cup product on $(\mathbb{C}^2)^{[n]}$ and a 'filtered' convolution product on S_n has been subsequently established by Lehn-Sorger [**LS**] and Vasserot [**Vas**]. Very recently Ruan (cf. [**Ruan**]) observed that such a product structure on $R(S_n)$ coincides with the orbifold cup product (introduced in [**CR**]) for the symmetric product $\mathbb{C}^{2n}/S_n$. This raises the interesting question how to relate the cohomology ring structure of Hilbert schemes $X^{[n]}$ to the orbifold cohomology ring structure of the symmetric products X^n/S_n or more generally of the wreath product orbifolds Y^n/Γ_n [3].

In the case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, Y is $\mathbb{C}^2$, and X is the minimal resolution of singularities $\mathbb{C}^2//\Gamma$, the connection between Hilbert scheme $(\mathbb{C}^2//\Gamma)^{[n]}$ and the wreath products Γ_n is very intriguing, cf. [**Wa2**]. The natural morphism $\tau_n : (\mathbb{C}^2//\Gamma)^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$ is a semismall crepant resolution of singularities. In [**Wa2**], we singled out a distinguished subvariety $Y_{\Gamma,n}$ of the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ which is a crepant resolution of $\mathbb{C}^{2n}/\Gamma_n$, where N is the order of Γ. It can be shown (cf. the Appendix) that $Y_{\Gamma,n}$ affords a description of quiver varieties in the sense of Nakajima [**Na1, Na4**] and it is diffeomorphic to $(\mathbb{C}^2//\Gamma)^{[n]}$. We established a morphism from the so-called Γ_n-Hilbert scheme $\mathbb{C}^{2n}//\Gamma_n$ in $\mathbb{C}^{2n}$ to the $Y_{\Gamma,n}$, and conjectured it is an isomorphism [**Wa2**] (we refer the reader to [**Nra,**

[2]There has appeared further interesting development in the symmetric group case by Lascoux and Thibon [**L-T**].

[3]In [**LS2**], Lehn and Sorger have constructed the cohomology ring on $X^{[n]}$ in terms of symmetric group S_n and the cohomology ring $H^*(X)$, when the surface X has numerically trivial canonical class. The connection between Lehn-Sorger's construction and Chen-Ruan's orbifold cup product is further clarified in [**FG**].

Rei1, BKR] for more study of the G-Hilbert schemes in different situations). This conjecture for $n = 1$ is a theorem of Ito-Nakamura [**INr1**] and Ginzburg-Kapranov (unpublished) which provides a uniform construction of the minimal resolutions of simple singularities in terms of Hilbert schemes of points on $\mathbb{C}^2$. On the other hand, when Γ is trivial it is a remarkable theorem of Haiman [**Hai2**] which is used to settle the $n!$ conjecture of Garsia-Haiman and the Macdonald polynomial positivity conjecture [**Mac, Hai1**]. The combinatorial implications in our general case (i.e. Γ nontrivial including Γ cyclic) along this line will be pursued elsewhere.

The layout of this paper is as follows. In Sect. 2, we review the generating function for the Betti numbers of Hilbert schemes, the constructions of Heisenberg algebra and Virasoro algebra in the framework of Hilbert schemes. We also review results on the cohomology ring structure of Hilbert schemes. In Sect. 3, we explain the constructions of Heisenberg algebra and Virasoro algebra in the framework of wreath products. We indicate how to develop further to obtain a group-theoretic realization of the vertex representations of affine and toroidal Lie algebras etc. In Sect. 4, we exhibit the interplay between the Hilbert schemes and the wreath products. We discuss the relations between $(\mathbb{C}^2//\Gamma)^{[n]}$ and the Γ_n-Hilbert scheme $\mathbb{C}^{2n}//\Gamma_n$. We also review the precise correspondence between the cup product of $H^*((\mathbb{C}^2)^{[n]})$ and the 'filtered' convolution product of S_n. In the Appendix, we present the description of $Y_{\Gamma,n}$ as a quiver variety. In the end, we provide a dictionary table to make a comparison between Hilbert schemes and wreath products. We expect such a table will be greatly extended in the future.

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Note added. There has been much further development in closely related topics recently. We refer the reader to [**QW**] and the references therein for detail.

2. Algebraic structures behind Hilbert schemes

2.1. Hilbert schemes $X^{[n]}$. Let X be a smooth projective surface over $\mathbb{C}$, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in the Hilbert scheme $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ be the corresponding sheaf of ideals. For a point $x \in X$, let ξ_x be the component of ξ supported at x and $I_{\xi,x} \subset \mathcal{O}_{X,x}$ be the stalk of I_ξ at x. A theorem of Fogarty [**Fog**] says that $X^{[n]}$ is smooth. In $X^{[n]} \times X$, we have the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\} \subset X^{[n]} \times X.$$

By sending an element in $X^{[n]}$ to its support, we obtain the Hilbert-Chow morphism $\pi_n : X^{[n]} \to X^n/S_n$, which is a resolution of singularities.

For example, when $n = 2$, an element of $X^{[2]}$ may be a pair of two distinct points in X, or a pair consisting one point in X together with a tangent direction at this point. The situation becomes much more complicated for $n > 2$.

2.2. The Betti numbers of Hilbert schemes. Ellingsrud and Strømme [**ES1**] first calculated the Betti numbers for the Hilbert scheme $X^{[n]}$ when X is the projective plane, the affine plane, or a rational ruled surface. They used the toric action on these surfaces in an essential way, and thus their method cannot be

extended to more general surfaces. Using the Weil conjecture proved by Deligne, Göttsche [**Got1, Got2**] calculated the Betti numbers for $X^{[n]}$ associated to an arbitrary (quasi-)projective surface X. Moreover, he presented the solution in a beautiful generating function which indicates one should study the cohomology groups of $X^{[n]}$ for all $n \geq 0$ together.

Let us denote by $H^*(-)$ the cohomology group with complex coefficient, denote by $b_i(-)$ the i-th Betti number, and define the poincaré polynomial as

$$P_t(X) = \sum_{i \geq 0} t^i b_i(X).$$

THEOREM 2.1. [**Got1, Got2**] *Let X be a smooth quasi-projective surface. The generating function in a variable q of the Poincaré polynomials of the Hilbert scheme $X^{[n]}$ is given by*

$$\sum_{n=0}^{\infty} P_t(X^{[n]}) q^n = \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(X)}(1+t^{2m+1}q^m)^{b_3(X)}}{(1-t^{2m-2}q^m)^{b_0(X)}(1-t^{2m}q^m)^{b_2(X)}(1-t^{2m+2}q^m)^{b_4(X)}}$$

In particular, the above formula implies the following generating function for the dimension of the total cohomology group $H^*(X^{[n]})$:

$$\sum_{n=0}^{\infty} \dim H^*(X^{[n]}) q^n = \prod_{m=1}^{\infty} \frac{(1+q^m)^{h^{odd}(X)}}{(1-q^m)^{h^{ev}(X)}} \tag{2.1}$$

where $h^{odd}(X) = b_1(X) + b_3(X)$, and $h^{ev}(X) = b_0(X) + b_2(X) + b_4(X)$. We will denote

$$\mathbb{H}_X = \bigoplus_{n \geq 0} H^*(X^{[n]}).$$

As remarked by de Cataldo and Migliorini [**dCM**], the formulas above remain valid for any complex surface if one replaces the Hilbert scheme by a suitable notion called the Douady space.

2.3. Heisenberg algebra and Hilbert schemes. Vafa and Witten [**VW**] observed that the formula (2.1) coincides with the character formula of a Fock space of a Heisenberg (super)algebra (i.e free bosons/fermions) associated to the lattice $H^*(X, \mathbb{Z})/tor$. This motivated them to conjecture that there indeed exists such a Heisenberg algebra acting on the direct sum $\mathbb{H}_X = \sum_{n \geq 0} H^*(X^{[n]})$ of the cohomology groups of the Hilbert schemes $X^{[n]}$. For the sake of simplicity, we refrain ourselves to the case when X is a projective surface when we recall the construction of Nakajima [**Na2**] (also compare [**Gro**]) below.

For $n \geq 0$ and $\ell \geq 0$, we define $Q^{[n+\ell,n]} \subset X^{[n+\ell]} \times X \times X^{[n]}$ to be the closed subset

$$\{(\xi, x, \eta) \in X^{[n+\ell]} \times X \times X^{[n]} \,|\, \xi \supset \eta \text{ and } \mathrm{Supp}(I_\eta/I_\xi) = \{x\}\}.$$

In particular, $Q^{[n,n]} = \emptyset$.

For $n \in \mathbb{Z}$, we define a linear map $\mathfrak{q}_n : H^*(X) \mapsto \mathrm{End}(\mathbb{H}_X)$ as follows. When $n \geq 0$, the linear operator $\mathfrak{q}_n(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ is defined by

$$\mathfrak{q}_n(\alpha)(a) = \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a).$$

for all $a \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively.

The space $\mathbb{H}_X = \oplus_{n,k} H^k(X^{[n]})$ actually carries a bi-degree provided by the conformal weight n and the cohomology degree k respectively. A non-degenerate super-symmetric bilinear form $(,)$ on $\mathbb{H}_X$ is induced from the standard one on $H^*(X^{[n]})$. For a homogeneous linear operator $\mathfrak{f} \in \mathrm{End}(\mathbb{H})$ of bi-degree (ℓ, m), we can define its *adjoint* $\mathfrak{f}^\dagger \in \mathrm{End}(\mathbb{H})$ by

$$(\mathfrak{f}(\alpha), \beta) = (-1)^{m|\alpha|} \cdot (\alpha, \mathfrak{f}^\dagger(\beta))$$

where $|\alpha| = s$ for $\alpha \in H^s(X)$.

When $n < 0$, define the operator $\mathfrak{q}_n(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ by

$$\mathfrak{q}_n(\alpha) = (-1)^n \cdot \mathfrak{q}_{-n}(\alpha)^\dagger.$$

The operator $\mathfrak{q}_n(\alpha)$ can be alternatively defined by switching the role of $\tilde{p}_1$ and $\tilde{p}_2$ in the definition of $\mathfrak{q}_{-n}(\alpha)$.

THEOREM 2.2. [**Na3**] *Let X be a smooth projective surface. The operators $\mathfrak{q}_n(\alpha)$ ($n \in \mathbb{Z}$, $\alpha \in H^*(X)$) acting on $\mathbb{H}_X$ satisfy the Heisenberg algebra commutation relations:*

$$[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n \cdot \delta_{n+m} \cdot \int_X (\alpha\beta) \cdot \mathrm{Id}_{\mathbb{H}}.$$

Moreover, the space $\mathbb{H}_X = \sum_{n \geq 0} H^*(X^{[n]})$ is an irreducible representation of this Heisenberg algebra.

The commutator above is understood as a supercommutator when both α and β are cohomology classes of odd degrees. Indeed the theorem is valid for an arbitrary quasi-projective surface X with some appropriate modification. We refer to [**Na3**] for a proof.

2.4. Virasoro algebra and Hilbert schemes. Define the linear operator $\mathfrak{d} \in \mathrm{End}(\mathbb{H}_X)$ to be the cup product with $c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n}) = -[\partial X^{[n]}]/2$ in $\mathbb{H}_n$ for each n. Here p_1 is the projection of $X^{[n]} \times X$ to $X^{[n]}$, $\partial X^{[n]}$ is the boundary of $X^{[n]}$ consisting of all $\xi \in X^{[n]}$ with $|\mathrm{Supp}(\xi)| < n$, and $c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n})$ denotes the first Chern class of the rank-n bundle $p_{1*}\mathcal{O}_{\mathcal{Z}_n}$. Below K_X and $c_2(X)$ stand for the canonical divisor and the second Chern class of X respectively.

THEOREM 2.3. [**Lehn**] *Let X be a smooth projective surface. The commutators of $\mathfrak{d}$ and the Heisenberg generators $\mathfrak{q}_n(\alpha)$ provide us the Virasoro generators (acting on $\mathbb{H}_X$):*

$$[\mathfrak{d}, \mathfrak{q}_n(\alpha)] = n \cdot \mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2} \mathfrak{q}_n(K_X\alpha),$$

where the operators $\mathfrak{L}_n(\alpha)$ satisfy the Virasoro algebra commutation relations:

$$[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m) \cdot \mathfrak{L}_{n+m}(\alpha\beta) - \frac{n^3 - n}{12} \cdot \delta_{n+m} \cdot \int_X (c_2(X)\alpha\beta) \cdot \mathrm{Id}_{\mathbb{H}_X}.$$

Let us recall some elementary construction in vertex algebras, cf. [**Bor1, FLM, Kac2**]. Let $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{n-\Delta}$ be a vertex operator of conformal weight Δ,

that is, a generating function in a formal variable z where $a_{(n)}$ is an operator acting on $\mathbb{H}_X$ such that $a_{(n)}(H^*(X^{[m]})) \subset \mathbb{H}^*(X^{[m+n]})$. Put

$$a_+(z) = \sum_{n>0} a_{(n)} z^{n-\Delta} \qquad \text{and} \qquad a_-(z) = \sum_{n\le 0} a_{(n)} z^{n-\Delta}$$

(note our unusual sign convention here on vertex operators). If $b(z)$ is another vertex operator, we define a new vertex operator, which is called *the normally ordered product* of $a(z)$ and $b(z)$, to be:

$$: a(z)b(z) := a_+(z)b(z) + (-1)^{ab} b(z)a_-(z)$$

where $(-1)^{ab}$ is -1 if both $a(z)$ and $b(z)$ are odd and 1 otherwise. Inductively we can define the normally ordered product of k vertex operators from right to left by

$$: a_1(z)a_2(z)\cdots a_k(z) : = : a_1(z)(: a_2(z)\cdots a_k(z) :) : .$$

For $\alpha \in H^*(X)$, we define a vertex operator $\alpha(z)$ by putting

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \mathfrak{q}_n(\alpha) z^{n-1}.$$

Let $k \ge 1$, $n \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Let $\delta_{k*} : H^*(X) \to H^*(X^k) \cong H^*(X)^{\otimes k}$ be the linear map induced by the diagonal embedding $\delta_k : X \to X^k$, and set

$$\delta_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \ldots \otimes \alpha_{j,k}.$$

We define the operator $W_n^k(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ to be the coefficient of z^{n-k} in the vertex operator

$$\frac{1}{k!} \cdot (\delta_{k*}\alpha)(z) = \frac{1}{k!} \cdot \sum_j : \alpha_{j,1}(z) \cdots \alpha_{j,k}(z) : .$$

Note that $W_n^1(\alpha)$ coincides with the Heisenberg generator $\mathfrak{q}_n(\alpha)$, and $W_n^2(\alpha)$ provides us the Virasoro generator $\mathfrak{L}_n(\alpha)$, cf. [**Lehn**].

It is proved in [**LQW1**] that if the canonical class K_X is numerically trivial then

$$\mathfrak{d} = -W_0^3(1_X). \tag{2.2}$$

It is very interesting to compare with Eqn. (3.3).

Recall that $\mathcal{Z}_n$ is the universal codimension 2 subscheme of $X^{[n]} \times X$, and p_1 and p_2 are the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively. Given a line bundle (i.e. a locally free sheaf of rank 1) L in X, one can define

$$L^{[n]} := p_{1*}((\mathcal{O}_{\mathcal{Z}_n}) \otimes p_2^* L).$$

Since p_1 is a flat finite morphism of degree n, $L^{[n]}$ is a locally free sheaf on $X^{[n]}$ of rank n. Using the operator $\mathfrak{d}$ effectively, Lehn proved a beautiful formula for the total Chern class of $L^{[n]}$. This formula was conjectured earlier by Göttsche based on some closely related formula of Grojnowski and Nakajima (see [**Na3**]).

THEOREM 2.4. [**Lehn**] *Let L be a line bundle on X. Then*

$$\sum_{n\ge 0} c(L^{[n]}) z^n \quad = \quad \exp\left(\sum_{n\ge 1} (-1)^{n-1} \frac{1}{n}\, \mathfrak{q}_n(c(L)) z^n \right). \tag{2.3}$$

2.5. The cohomology ring structure of $X^{[n]}$. For $\gamma \in H^s(X)$ and $n \geq 0$, let $G_i(\gamma, n)$ be the homogeneous component in $H^{s+2i}(X^{[n]})$ of

$$G(\gamma, n) = p_{1*}(\mathrm{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^*\mathrm{td}(X) \cdot p_2^*\gamma) \in H^*(X^{[n]}).$$

Here and below we omit the Poincaré duality used to switch a homology class to a cohomology class and vice versa. The following theorem has been established by Li, Qin and the author.

THEOREM 2.5. [**LQW1**] *Let X be a smooth projective surface. The cohomology ring of the Hilbert scheme $X^{[n]}$ is generated by the classes $G_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$.*

Let us briefly comment on the proof of the above theorem. We took the viewpoint effectively that for a fixed γ the cup products with $G_i(\gamma, n)$ for all n should be treated as a single operator $\mathfrak{G}_i(\gamma)$ acting on $\mathbb{H}_X = \oplus_n H^*(X^{[n]})$. Exploring relations between the operators $\mathfrak{G}_i(\gamma)$ and the Heisenberg operators $\mathfrak{q}_n(\alpha)$, we found some surprising connections between $\mathfrak{G}_i(\gamma)$ and the zero-mode of a vertex operator $W^{i+2}(\gamma)$. This allowed us to finish the proof by using an induction.

A set of ring generators for $H^*(X^{[n]})$ when X is $\mathbb{P}^2$ (which is easily shown to be equivalent to the set given in the above Theorem) or $\mathbb{C}^2$ was first found by Ellingsrud and Strømme [**ES2**]. Lehn's new proof [**Lehn**] using the Heisenberg algebra for this result when $X = \mathbb{C}^2$ has been very inspiring for our approach in [**LQW1**]. On the other hand, the approach of Ellingsrud and Strømme has been extended to other rational surfaces and ruled surfaces in [**Bea**], and to $K3$ surfaces by Markman [**Mar**].

The generators $G_i(\gamma, n)$ has the advantage of having certain nice commutation relations with the Heisenberg generators. But their geometric meaning is not always very clear. A new set of ring generators for $X^{[n]}$ associated to an arbitrary projective surface was found in [**LQW2**] which admits simple geometric interpretation.

Let us introduce some notations. Let $|0\rangle$ denote the element 1 of $H^0(X^{[0]}) = \mathbb{C}$. For $0 \leq i < n$ and $\gamma \in H^i(X)$, define a cohomology class $B_i(\gamma, n) \in H^{s+2i}(X^{[n]})$ by putting

$$B_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot \mathfrak{q}_{i+1}(\gamma)\mathfrak{q}_1(1_X)^{n-i-1}|0\rangle.$$

Note that these are among the simplest cohomology classes in $H^*(X^{[n]})$ in geometric terms. Indeed, $B_0(1_X, n) = n \cdot 1_{X^{[n]}}$. If either $i > 0$ or $\gamma \in H^s(X)$ with $s > 0$, then one can easily show that the Poincaré dual of $B_i(\gamma, n)$ is the homology class represented by the closed subset:

$$\{\, \xi \in X^{[n]} \,|\, \exists\, x \in \Upsilon \text{ with } \ell(\xi_x) \geq i + 1 \,\}$$

where Υ is a homology cycle of X representing the Poincaré dual of γ, and ξ_x is the component of ξ such that ξ_x is supported at x.

THEOREM 2.6. [**LQW2**] *Let X be a smooth projective surface. The cohomology ring of the Hilbert scheme $X^{[n]}$ is generated by the classes $B_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$.*

The proof of this theorem is based on establishing certain relations between $B_i(\gamma, n)$ and $G_j(\alpha, n)$ introduced earlier and an induction. We also took the viewpoint that for a fixed γ the cup products with $B_i(\gamma, n)$ for all n should be treated as a single operator $\mathfrak{B}_i(\gamma)$ acting on $\mathbb{H}_X = \oplus_n H^*(X^{[n]})$.

3. Algebraic structures behind wreath products

3.1. The wreath product Γ_n. Let Γ be a finite group and denote by Γ_* the set of conjugacy classes of Γ. Denote by $[g]$ the conjugacy class of $g \in \Gamma$. Denote by ζ_c the order of the centralizer of an element g in the conjugacy class c of Γ. Let $\Gamma^n = \Gamma \times \ldots \times \Gamma$ be the direct product of n copies of Γ. The symmetric group S_n acts on the product group Γ^n by permuting the n factors: $s(g_1, \ldots, g_n) = (g_{s^{-1}(1)}, \ldots, g_{s^{-1}(n)})$. The *wreath product* Γ_n is defined to be the semi-direct product $\Gamma^n \rtimes S_n$ of Γ^n and S_n, namely the multiplication on Γ_n is given by $(g, s)(h, t) = (g.s(h), st)$, where $g, h \in \Gamma^n, s, t \in S_n$. Note when Γ is the trivial one-element group the wreath product Γ_n reduces to S_n, and when Γ is $\mathbb{Z}_2$ the wreath product Γ_n is the hyperoctahedral group, the Weyl group of type B.

We denote by $R_{\mathbb{Z}}(\Gamma)$ the lattice with a basis given by the irreducible characters of Γ. Then $R(\Gamma) := R_{\mathbb{Z}}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$ can be regarded as the representation ring of Γ or the space of class functions on Γ. Denote by

$$\mathbb{R}_\Gamma = \bigoplus_{n=0}^{\infty} R(\Gamma_n).$$

The usual bilinear form on $R(\Gamma)$ is defined by

$$\langle f, g \rangle \equiv \langle f, g \rangle_\Gamma = \frac{1}{|\Gamma|} \sum_{x \in \Gamma} f(x) g(x^{-1}),$$

and the set Γ^* of the irreducible characters of Γ form an orthonormal basis for $R(\Gamma)$.This provides us a bilinear form $\langle -, - \rangle$ on $\mathbb{R}_\Gamma$.

It is well known that the conjugacy classes of S_n are parameterized by the partitions of n. The conjugacy classes of a general wreath product can be described as follows, (cf. e.g. [**Mac**]). Given $a = (g, s) \in \Gamma_n$ where $g = (g_1, \ldots, g_n)$, we write $s \in S_n$ as a product of disjoint cycles: if $z = (i_1, \ldots, i_r)$ is one of them, the *cycle-product* $g_{i_r} g_{i_{r-1}} \ldots g_{i_1}$ of a corresponding to the cycle z is determined by g and z up to conjugacy. For each $c \in \Gamma_*$ and each integer $r \geq 1$, let $m_r(c)$ be the number of r-cycles in s whose cycle-product lies in c. Denote by $\rho(c)$ the partition having $m_r(c)$ parts equal to r ($r \geq 1$) and denote by $\rho = (\rho(c))_{c \in \Gamma_*}$ the corresponding partition-valued function on Γ_*. Note that $||\rho|| := \sum_{c \in \Gamma_*} |\rho(c)| = \sum_{c \in \Gamma_*, r \geq 1} r m_r(c) = n$, where $|\rho(c)|$ is the size of the partition $\rho(c)$. Thus we have defined a map from Γ_n to $\mathcal{P}_n(\Gamma_*)$, the set of partition-valued function $\rho = (\rho(c))_{c \in \Gamma_*}$ on Γ_* such that $||\rho|| = n$. The function ρ or the data $\{m_r(c)\}_{r,c}$ is called the *type* of $a = (g, s) \in \Gamma_n$. Denote $\mathcal{P}(\Gamma_*) = \sum_{n \geq 0} \mathcal{P}_n(\Gamma_*)$. It can be shown that two elements in Γ_n are conjugate to each other if and only if they have the same type.

3.2. Heisenberg algebra and wreath products. Denote by $c_n (c \in \Gamma_*)$ the conjugacy class in Γ_n of elements $(x, s) \in \Gamma_n$ such that s is an n-cycle and the cycle product of x is c. Denote by $\sigma_n(c)$ the class function on Γ_n which takes value $n\zeta_c$ (i.e. the order of the centralizer of an element in the class c_n) on elements in the class c_n and 0 elsewhere. Given $\gamma \in R(\Gamma)$, we denote by $\sigma_n(\gamma)$ the class function on

Γ_n which takes value $n\gamma(c)$ on elements in the class $c_n, c \in \Gamma_*$, and 0 elsewhere. In the symmetric group (i.e. Γ trivial) case, the definition of $\sigma_n(1)$ is much simplified.

We define $p_n(\gamma), n > 0$ to be a map from $\mathbb{R}_\Gamma$ to itself by the following composition:

$$R(\Gamma_m) \overset{\sigma_n(\gamma)\otimes}{\longrightarrow} R(\Gamma_n) \bigotimes R(\Gamma_m) \overset{Ind}{\longrightarrow} R(\Gamma_{n+m}).$$

We define $p_{-n}(\gamma), n > 0$ to be the adjoint of $p_n(\gamma)$ with respect to the bilinear form $\langle -, - \rangle$ on $\mathbb{R}_\Gamma$, or equivalently a map from $\mathbb{R}_\Gamma$ to itself as the composition

$$R(\Gamma_m) \overset{Res}{\longrightarrow} R(\Gamma_n) \bigotimes R(\Gamma_{m-n}) \overset{\langle \sigma_n(\gamma), \cdot \rangle}{\longrightarrow} R(\Gamma_{m-n}).$$

The following theorem is proved by using Mackey's theorem on the restriction to a subgroup of an induced character. Indeed it is a special case of a theorem established in [**Wa1**] in a more general (i.e. equivariant K groups) setup.

THEOREM 3.1. [**Wa1**] *The operators $p_n(\gamma)$, $n \in \mathbb{Z}, \gamma \in R(\Gamma)$, satisfy the Heisenberg commutation relations:*

$$[p_m(\gamma), p_n(\gamma')] = -m\delta_{m,-n}\langle \gamma, \gamma' \rangle C, \quad \gamma, \gamma' \in R(\Gamma).$$

Moreover, $\mathbb{R}_\Gamma$ is an irreducible representation of the Heisenberg algebra.

It follows that $\mathbb{R}_\Gamma$ is isomorphic to the symmetric algebra generated by $p_m(\gamma)$ where $\gamma \in \Gamma^*, n > 0$. The construction of Heisenberg algebra above is intimately related to the Hopf algebra structure on $\mathbb{R}_\Gamma$ given by Zelevinsky [**Zel**]. One may choose a different natural basis of the Heisenberg algebra parameterized by $n \in \mathbb{Z}, c \in \Gamma_*$ (cf. [**FJW1**]):

$$p_n(c) = \sum_{\gamma \in \Gamma^*} \gamma(c^{-1}) p_n(\gamma),$$

where c^{-1} denotes the conjugacy class $\{g \in \Gamma | \gamma^{-1} \in c\}$.

Given a representation V of Γ with character γ, the natural actions of Γ^n and S_n on n-th outer tensor product $V^{\otimes n}$ of V can be combined into an action of the wreath product Γ_n. We denote the Γ_n-character of $V^{\otimes n}$ by $\eta_n(\gamma)$. Denote by ε_n the (1-dimensional) sign representation of Γ_n on which Γ^n acts trivially while S_n acts as sign representation. We denote by $\varepsilon_n(\gamma) \in R(\Gamma_n)$ the character of the tensor product of ε_n and $V^{\otimes n}$.

The following proposition can be found in [**Wa1**] in a more general (i.e. equivariant K-group) setup. These equalities here are well known for $\Gamma \in \Gamma^*$, cf. [**Mac, FJW1**]. We remark that this type of formulas together with a parallel formula (2.3) often appear as half of a vertex operator in vertex algebra literature.

PROPOSITION 3.2. For any $\gamma \in R(\Gamma)$, we have

$$\sum_{n\geq 0} \mathrm{ch}(\eta_n(\gamma)) z^n = \exp\left(\sum_{n \geq 1} \frac{1}{n} a_{-n}(\gamma) z^n \right),$$

$$\sum_{n\geq 0} \mathrm{ch}(\varepsilon_n(\gamma)) z^n = \exp\left(\sum_{n \geq 1} (-1)^{n-1} \frac{1}{n} a_{-n}(\gamma) z^n \right). \tag{3.1}$$

3.3. McKay correspondence and wreath products. In the case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, it is pointed out in [**Wa1**] that one can indeed develop the above construction further to realize the basic vertex representations of affine Lie algebras of ADE types in a group theoretic manner. This has subsequently been fully carried out in collaboration with I. Frenkel and Jing [**FJW1**].

The classification of finite subgroups of $SL_2(\mathbb{C})$ is of course well known. First it is clear that finite subgroups of $SL_2(\mathbb{C})$ are indeed finite subgroups of SU_2. Note that SU_2 is a double cover of SO_3, so the classification is essentially reduced to the classification of finite subgroups of SO_3, which consists of the symmetric groups of regular polyhedra (i.e. tetrahedral, octahedral and icosahedral groups) and the symmetric groups of certain two dimensional 'degenerate regular polyhedra' (i.e. cyclic and dihedral groups). This results into a complete list of finite subgroups of $SL_2(\mathbb{C})$: the cyclic, binary dihedral, tetrahedral, octahedral and icosahedral groups. (Binary here means a double cover).

Denote by γ_0 the trivial character of Γ and Q the two-dimensional defining representation of Γ in $\mathbb{C}^2$. Let $\xi = 2\gamma_0 - Q$. Introduce the *weighted* bilinear form (cf. [**FJW1**]) on $R(\Gamma)$ by

$$\langle f, g\rangle_\xi = \langle \xi \otimes f, g\rangle_\Gamma, \quad f, g \in R(\Gamma).$$

Then McKay's observation [**McK**] simply says the lattice $(R_{\mathbb{Z}}(\Gamma), \langle -, -\rangle_\xi)$ is the affine root lattice of ADE types, and this gives rise to a bijection between the set of finite subgroups of $SL_2(\mathbb{C})$ and the set of affine root lattice (or equivalently affine Dynkin diagrams) of ADE types. For example, cyclic corresponds to type A.

Recall $\eta_n(\xi)$ is defined to be a virtual character in $R(\Gamma_n)$. We define a weighted bilinear form on $R(\Gamma_n)$ by

$$\langle f, g\rangle_\xi = \langle \eta_n(\xi) \otimes f, g\rangle_{\Gamma_n}, \quad f, g \in R(\Gamma_n),$$

and thus a bilinear form $\langle -, -\rangle_\xi$ on $\mathbb{R}_\Gamma = \oplus_n R(\Gamma_n)$.

Let us digress for a moment. Given an integral lattice L, we can construct a Heisenberg algebra H_L which is the central extension of the loop algebra $\mathfrak{h}\otimes\mathbb{C}[t, t^{-1}]$, where $\mathfrak{h}$ denotes $L \otimes_{\mathbb{Z}} \mathbb{C}$. One can endow a canonical structure of an irreducible representation of H_L on the symmetric algebra $\mathrm{Sym}(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$. Following the usual consideration for a lattice vertex algebra [**Bor1, FLM, Kac2**], we introduce the space $V_L = \mathrm{Sym}(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \bigotimes \mathbb{C}[L]$, where $\mathbb{C}[L]$ denotes the group algebra of the lattice (i.e. free abelian group) L. A celebrated theorem of Frenkel-Kac [**FK**] and Segal [**Seg1**] says that if L is a simple root lattice of ADE type then V_L provides an realization of the basic representation of the affine Kac-Moody algebra (which is the central extension of a loop algebra associated to the lattice L), cf. [**Kac1**]. The construction is based on the action of the Heisenberg algebra H_L and vertex operators. This construction has been extended in [**MRY**] to realize the vertex representation of a toroidal Lie algebra (which is a central extension of a double loop algebra associated to L) in the case when L is an affine root lattice.

We now introduce the following space

$$V_\Gamma = \mathbb{R}_\Gamma \bigotimes \mathbb{C}[R_{\mathbb{Z}}(\Gamma)],$$

where $\mathbb{C}[R_{\mathbb{Z}}(\Gamma)]$ is the group algebra of the lattice $(R_{\mathbb{Z}}(\Gamma), \langle -, -\rangle_\xi)$.

We formulate informally a main theorem in [**FJW1**] below, and refer the reader to the original paper for more detail. Our constructions can be regarded as a new form of McKay correspondence: starting from a finite subgroup Γ of $SL_2(\mathbb{C})$, we

realize the vertex representation of an affine and toroidal Lie algebra whose Dynkin diagram corresponds to Γ in the classical way.

THEOREM 3.3. *For a finite subgroup Γ of $SL_2(\mathbb{C})$, the space V_Γ provides a group-theoretic realization of the vertex representation of a toroidal Lie algebra associated to the affine root lattice $(R_{\mathbb{Z}}(\Gamma), \langle -, - \rangle_\xi)$. The generators for the toroidal Lie algebra are given in terms of vertex operators and afford a natural group-theoretic interpretation in terms of certain induction/restriction functors.*

One can identify a distinguished subspace $\overline{V}_\Gamma$ of V_Γ (which is, roughly speaking, the subspace to which the trivial character γ_0 of Γ makes no contribution) in which we realize in a group-theoretic manner the basic vertex representation of an affine Lie algebra.

Note that the action of Γ_n on $\mathbb{C}^{2n}$ induces a natural degree-preserving Γ_n-action on the exterior algebra $\Lambda^*(\mathbb{C}^{2n}) = \oplus_{i=0}^{2n} \Lambda^i(\mathbb{C}^{2n})$. It we understand $\mathbb{C}^{2n}$ as the Γ_n-equivariant vector bundle over a point, then $\Lambda^*(\mathbb{C}^{2n})$ is exactly the associated Koszul-Thom complex. Note that the virtual Γ-character ξ can be rewritten as $\sum_{i=0}^{2}(-1)^i \Lambda^i(\mathbb{C}^2)$.

PROPOSITION 3.4. [**Wa2**] The virtual Γ_n-character $\sum_{i=0}^{2n}(-1)^i \Lambda^i(\mathbb{C}^{2n})$ coincides with the virtual Γ_n-character $\eta_n(\xi)$, where the Γ-virtual character ξ is given by $\sum_{i=0}^{2}(-1)^i \Lambda^i(\mathbb{C}^2)$.

REMARK 3.5. We denote by $D_{\Gamma_n}(\mathbb{C}^{2n})$ the bounded derived category of Γ_n-equivariant coherent sheaves on $\mathbb{C}^{2n}$. We denote by $D^0_{\Gamma_n}(\mathbb{C}^{2n})$ the full subcategory of $D_{\Gamma_n}(\mathbb{C}^{2n})$ consisting of objects whose cohomology sheaves are concentrated on the origin of $\mathbb{C}^{2n}$, and by $K^0_{\Gamma_n}(\mathbb{C}^{2n})$ the corresponding K-group. A natural bilinear form on $K^0_{\Gamma_n}(\mathbb{C}^{2n})$ uses the Γ_n-virtual character $\sum_{i=0}^{2n}(-1)^i \Lambda^i(\mathbb{C}^{2n})$. It follows from the above proposition that there exists a canonical isometric isomorphism between $K^0_{\Gamma_n}(\mathbb{C}^{2n})$ endowed with this bilinear form and $R_{\mathbb{Z}}(\Gamma_n)$ endowed with the weighted bilinear form, cf. [**Wa2**].

The realization of the homogeneous vertex representation of an affine algebra on $\overline{V}_\Gamma$ above can be regarded as a counterpart of a realization in terms of the middle dimensional cohomology group of the Hilbert schemes $X^{[n]}$ when X is the minimal resolution $\mathbb{C}^2//\Gamma$ of the simple singularity $\mathbb{C}^2/\Gamma$, cf. [**Na3**]. However the toroidal Lie algebras do not appear in such a cohomology group setup since one cannot put the cohomology groups of different degrees of $X^{[n]}$ on the same footing. We believe if one can construct the Heisenberg algebra on the direct sum of the K-groups of $(\mathbb{C}^2//\Gamma)^{[n]}$ for all $n \geq 0$, then one would be able to obtain toroidal algebras as well. It would be very interesting to establish a direct isomorphism between these two constructions of toroidal Lie algebras.

By adding $\mathbb{C}^*$ to the picture appropriately and noting the representation ring of the algebraic representations of $\mathbb{C}^*$ can be identified with the polynomial ring $\mathbb{C}[q, q^{-1}]$, we [**FJW2**] obtained a group-theoretic realization of the Frenkel-Jing construction of quantum affine algebras and a construction of the quantum toroidal algebras by Ginzburg, Kapranov and Vasserot. We remark that the fact that the two-dimensional defining representation Q of Γ is reducible for Γ cyclic is reflected by the fact that the toroidal algebra of type A affords a *two*-parameter deformation. It would be interesting to obtain a realization of vertex representations of quantum toroidal algebras using the equivariant K-groups $K_{\mathbb{C}^*}((\mathbb{C}^2//\Gamma)^{[n]})$.

Using spin/projective representations of wreath products, we have realized in [**FJW3, JW**] the vertex representations of certain twisted affine and toroidal algebras.

3.4. Virasoro algebra and wreath products. Given $f, g \in \mathbb{C}[\Gamma]$, the *convolution* product on $\mathbb{C}[\Gamma]$ is defined by

$$(f * g)(x) = \sum_{y \in \Gamma} f(xy^{-1})g(y), \quad f, g \in \mathbb{C}[\Gamma], x \in \Gamma.$$

In particular if $f, g \in R(\Gamma)$, then so is $f * g$. It is well known that

$$\gamma' * \gamma = \frac{|\Gamma|}{d_\gamma} \delta_{\gamma,\gamma'} \gamma, \quad \gamma', \gamma \in \Gamma^*, \tag{3.2}$$

where d_γ is the *degree* of the irreducible character γ.

Denote by K_c the sum of all elements in a conjugacy class c. By abuse of notation, we also regard K_c the class function on Γ which takes value 1 on elements in the conjugacy class c and 0 elsewhere. It is clear that $K_c, c \in \Gamma_*$, form a basis of $R(\Gamma)$. The elements $K_c, c \in \Gamma_*$ actually form a linear basis of the center in the group algebra $\mathbb{C}[\Gamma]$.

Given $c \in \Gamma_*$, we denote by $K_i(c, n)$ the conjugacy class corresponding to the partition-valued function which maps the class c to the one-part partition $(i+1)$, c^0 to the partition (1^{n-i-1}) and other classes in Γ_* to 0. Let us define an operator $\mathfrak{d}_i(K_c)$ on $\mathbb{R}_\Gamma$ to be the convolution product with $K_i(c, n)$ in $R(\Gamma_n)$ for each n. The $K_i(c, n)$ is the counterpart of the $B_i(\gamma, n) \in H^*(X^{[n]})$ studied in Section 2. In the symmetric group case, I. Frenkel and I had much discussion on $K_i(c, n)$ in early 1999.

THEOREM 3.6. [**FW**] *The commutation relation between the operator $\mathfrak{d}_1(c)$ associated to a conjugacy class $c \in \Gamma_*$ and the Heisenberg algebra generator $p_n(\gamma)$ is given by*

$$[\mathfrak{d}_1(K_c), p_n(\gamma)] = \frac{n|\Gamma|^2 \gamma(c^{-1})}{\zeta_c d_\gamma^2} L_n(\gamma),$$

where the operators $L_n(\gamma)$ acting on the space $\mathbb{R}_\Gamma$ satisfy the Virasoro commutation relations:

$$[L_n(\gamma), L_m(\gamma')] = (n-m)\delta_{\gamma,\gamma'} L_{n+m}(\gamma) - \frac{n^3 - n}{12} \delta_{\gamma,\gamma'} \delta_{n,-m},$$

where γ, γ' lie in Γ^*.

We can extend the operator $\mathfrak{d}_1(K_c)$ linearly to any $a \in R(\Gamma)$. We remark that the following *transfer property* holds: $[\mathfrak{d}_1(a), p_n(b)]$, where $a, b \in R(\Gamma)$, does not depend on a and b individually but only on the product $a * b$. Similar transfer properties have been observed for the various geometric operators introduced in the framework of Hilbert schemes [**Lehn, LQW1, LQW2**], cf. Sect. 2.

In the symmetric group case, the operator $\mathfrak{d}_1 = \mathfrak{d}_1(c^0)$ has been studied by Goulden [**Gou**] for a totally different purpose and rediscovered in [**FW**] in the search for the Virasoro algebra. In this case we have

$$\begin{aligned} \mathfrak{d}_1 &= \frac{1}{6} Res_{z=0} : a(z)^3 : z^2 \\ &= \frac{1}{2} \sum_{n,m>0} (p_n p_m p_{-n-m} + p_{n+m} p_{-n} p_{-m}), \end{aligned} \tag{3.3}$$

where $: a(z)^3 :$ denotes the normally ordered product of $a(z) = \sum_{n\in\mathbb{Z}} p_n z^{-n-1}$ with itself for three times. In general, $\mathfrak{d}_1(c)$ has a similar expression as an operator acting on $\mathbb{R}_\Gamma$, cf. [**FW**] for more detail.

4. Interplay between Hilbert schemes and wreath products

4.1. Desingularization of the wreath product orbifolds. Now we assume that Y is a quasi-projective surface acted upon by a finite group Γ, and assume that a resolution of singularities $\tau : X \to Y/\Gamma$ is given. The action of the product group Γ^n and the symmetric group S_n on the direct product Y^n induces a natural action of the wreath product Γ_n on Y^n: for $a = ((g_1, \ldots, g_n), s)$, we let

$$a.(x_1, \ldots, x_n) = (g_1 x_{s^{-1}(1)}, \ldots, g_n x_{s^{-1}(n)})$$

where $x_1, \ldots, x_n \in Y$. It is observed in [**Wa1**] that the following commutative diagram

$$\begin{array}{ccc} X^{[n]} & \xrightarrow{\pi_n} & X^n/S_n \\ \Big\downarrow{\scriptstyle \tau_n} & & \Big\downarrow{\scriptstyle \tau_{(n)}} \\ Y^n/\Gamma_n & \xleftarrow{\cong} & (Y/\Gamma)^n/S_n \end{array} \tag{4.1}$$

defines a resolution of singularities $\tau_n : X^{[n]} \to Y^n/\Gamma_n$. Here we used the observation that the wreath product orbifold is exactly the symmetric product of an orbifold: $(Y/\Gamma)^n/S_n = (Y^n/\Gamma^n)/S_n = Y^n/(\Gamma^n \rtimes S_n) = Y^n/\Gamma_n$.

Remark 4.1. A direct sum $\mathcal{F}_\Gamma(Y)$ of equivariant K-groups $K_{\Gamma_n}(Y^n) \otimes \mathbb{C}$ for $n \geq 0$ is shown [**Wa1**] to have several nice algebraic structures. In particular, a Heisenberg algebra associated to the lattice $K_\Gamma(Y)$ modulo torsion, constructed in terms additive K-group maps, acts on $\mathcal{F}_\Gamma(Y)$ irreducibly. This specializes to the construction of Segal [**Seg1**] for Γ trivial, and specializes to the construction on $\mathbb{R}_\Gamma = \oplus_n R(\Gamma_n)$ in Sect. 3 when Y is a point.

4.2. The resolution $X^{[n]} \to Y^n/\Gamma_n$ is good. A guidline here [**Wa1**] is that whenever the resolution $\tau : X \to Y/\Gamma$ is 'good' in a suitable sense then the resolution $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ is also 'good'; alternatively, whenever a reasonable statement can be made relating the resolution X and the orbifold Y/Γ, then the corresponding statement is true relating the resolution $X^{[n]}$ and the orbifold Y^n/Γ_n.

For example, if $\tau : X \to Y/\Gamma$ is a crepant, respectively semismall, resolution of singularities, then we can show that $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ is also a crepant, respectively semismall, resolution of singularities.

Let us recall that in the study of orbifold string theory Dixon *et al* [**DHVW**] introduced the *orbifold Euler number* as

$$\chi(M, G) = \frac{1}{|G|} \sum_{g_1 g_2 = g_2 g_1} \chi(M^{g_1, g_2}),$$

where M is a smooth manifold acted by a finite group G, M^{g_1, g_2} denotes the common fixed point set of g_1 and g_2 and $\chi(\cdot)$ denotes the usual Euler number.

One can also define the orbifold Hodge numbers $h^{s,t}(M, G)$ if M is a compact complex manifold (cf. Zaslow [**Zas**]). For each conjugacy class $c = [g] \in G_*$, let $Y_1^g, \cdots, Y_{N_c}^g$ be the connected components of the fixed-point set Y^g. On the tangent

space to each point in Y_α^g, g acts as a diagonal matrix $\mathrm{diag}(e^{2\pi\sqrt{-1}\theta_1}, \cdots, e^{2\pi\sqrt{-1}\theta_d})$, where $0 \le \theta_i < 1$. The shift number $F_\alpha^g = \sum_{j=1}^d \theta_j$ is an integer if we further assume g acts on the tangent space by a matrix in $SL(n, \mathbb{C})$. The *orbifold Hodge numbers* of the orbifold M/G are defined to be

$$h^{s,t}(Y,G) = \sum_{c=[g]\in G_*} \sum_{\alpha_c=1}^{N_c} h^{s-F^c_{\alpha_c}, t-F^c_{\alpha_c}}(Y^g_{\alpha_c}/Z(g)),$$

where $Z(g)$ is the centralizer of g in G, and $h^{*,*}(Y^g_{\alpha_c}/Z(g))$ denotes the dimension of the space of invariants $H^{*,*}(Y^g_{\alpha_c})^{Z(g)}$. We define the orbifold virtual Hodge polynomial by $e(M,G;x,y) = \sum_{s,t}(-1)^{s+t}h^{s,t}(M,G)x^s y^t$. In particular when G is trivial, the orbifold Euler number, orbifold Hodge numbers and orbifold virtual Hodge polynomials reduce to the usual ones.

Now let us get back to our setup when X, Y are complex surfaces, and $\tau : X \to Y/\Gamma$ is a resolution of singularities. The Euler and Hodge numbers of Hilbert schemes were calculated in [**Got1, Got2, GS**] (also cf. [**Che**]). On the other hand, the orbifold Euler and Hodge numbers for the wreath product orbifolds were calculated in [**Wa1, WaZ**]. In terms of generating functions, we have

$$\begin{aligned}
\sum_{n=0}^\infty \chi(X^{[n]})q^n &= \prod_{m=1}^\infty (1-q^m)^{-\chi(X)}, \\
\sum_{n=0}^\infty \chi(Y^n, \Gamma_n)q^n &= \prod_{m=1}^\infty (1-q^m)^{-\chi(Y,\Gamma)}, \\
\sum_{n=1}^\infty e(X^{[n]}; x, y)q^n &= \prod_{r=1}^\infty \prod_{s,t} (1 - x^s y^t q^r (xy)^{(r-1)})^{(-1)^{s+t+1}h^{s,t}(X)}, \\
\sum_{n=1}^\infty e(Y^n, \Gamma_n; x, y)q^n &= \prod_{r=1}^\infty \prod_{s,t} (1 - x^s y^t q^r (xy)^{(r-1)})^{(-1)^{s+t+1}h^{s,t}(Y,\Gamma)}.
\end{aligned}$$

As observed in [**Wa1, WaZ**], an immediate consequence of the above formulas is that if the Euler (resp. Hodge) number of X is equal to the orbifold Euler (resp. Hodge) number of the orbifold Y/Γ, then the Euler (resp. Hodge) number of $X^{[n]}$ is equal to the orbifold Euler (resp. Hodge) number of the orbifold Y^n/Γ_n. To our best knowledge, the resolutions $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ cover all known higher dimensional examples which match the Euler and Hodge numbers between the resolution and the orbifold. It has been conjectured [**WaZ**] that a similar statement holds by substituting (orbifold) Hodge numbers with (orbifold) elliptic genera (compare [**DMVV**]). Recently Borisov and Libgober [**BL**] obtained a mathematical formulation for orbifold elliptic genera and verified a formula for the elliptic genera of Y^n/Γ_n conjectured in [**WaZ**]. As we can easily construct various examples of good resolutions $\tau : X \to Y/\Gamma$, our construction provides in a tautological way numerous higher dimensional examples of good resolutions.

Note that the guiding principle, when applied to the situation when Γ is trivial and X equals Y, simply says that the Hilbert-Chow morphism $X^{[n]} \to X^n/S_n$ is a 'good' resolution of singularities in every aspect, cf. [**HH, Got3, Na3, DMVV, Zhou**]. Another very interesting example is the $K3$ surfaces which provide good resolutions of the $\mathbb{Z}/2\mathbb{Z}$ quotient of abelian surfaces. The coincidence between the

corresponding Hodge numbers of Hilbert schemes and wreath product orbifolds when X is $K3$ has also been obtained independently by Bryan, Donagi, and Leung [**BDL**].

A very important example is as follows. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. We denote by $\tau : \mathbb{C}^2//\Gamma \to \mathbb{C}^2/\Gamma$ the minimal resolution of a simple singularity. By applying the above general construction to the case $Y = \mathbb{C}^2$ and $X = \mathbb{C}^2//\Gamma$, we obtain the following commutative diagram:

$$
\begin{array}{ccc}
(\mathbb{C}^2//\Gamma)^{[n]} & \xrightarrow{\pi_n} & (\mathbb{C}^2//\Gamma)^n/S_n \\
\Big\downarrow{\scriptstyle \tau_n} & & \Big\downarrow{\scriptstyle \tau_{(n)}} \\
\mathbb{C}^{2n}/\Gamma_n & \xleftarrow{\cong} & (\mathbb{C}^2/\Gamma)^n/S_n
\end{array} \tag{4.2}
$$

This example will be examined from a different viewpoint in the next subsection.

4.3. The Γ_n-Hilbert scheme $\mathbb{C}^{2n}//\Gamma_n$. Let Y be a smooth complex algebraic variety and let Γ be a finite subgroup of order N in the group of automorphisms $\mathrm{Aut}(Y)$. A regular Γ-orbit can be viewed as an element in the Hilbert scheme $Y^{[N]}$ of N points in Y. The Γ-*Hilbert scheme* (or Hilbert scheme of Γ-regular orbits in Y) $Y//\Gamma$ in Y is defined to be the closure of the set of regular Γ-orbits in $Y^{[N]}$ (cf. e.g. [**Nra, Rei1**]). There is an induced Γ-action on $Y^{[N]}$, and it is clear that $Y//\Gamma$ is a component of the set $Y^{[N],\Gamma}$ of Γ-fixed points in $Y^{[N]}$.

A theorem of Ito-Nakamura [**INr1, INr2**] (also observed by Ginzburg and Kapranov) says the Γ-Hilbert scheme $\mathbb{C}^2//\Gamma$ associated to a finite group Γ of $SL_2(\mathbb{C})$ is the minimal resolution of $\mathbb{C}^2/\Gamma$. So our notation for orbit Hilbert schemes cause no real confusion with the previous sections where $\mathbb{C}^2//\Gamma$ is used to denote the minimal resolution. We refer the reader to [**INr2**] for many other interesting connections with other branches of mathematics.

The Hilbert-Chow morphism from $Y^{[N]}$ to $Y^{(N)}$ induces a morphism $Y//\Gamma \to Y/\Gamma$. The notion of orbit Hilbert schemes is particularly important when the morphism $Y//\Gamma \to Y/\Gamma$ turns out to be a (crepant) resolution of singularities (cf. [**Rei1, Nra, BKR**] and references therein). But in practice it is a very difficult problem to determine whether the Γ-Hilbert scheme is smooth and even more difficult to have a good description of it.

Below we restrict our study to the setup of the diagram (4.2). This material is taken from [**Wa2**]. We denote by $Y_{\Gamma,n}$ the set of Γ-invariant ideals I in the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ such that the quotient $\mathbb{C}[x,y]/I$ is isomorphic to a direct sum R^n of n copies of the regular Γ-representation R as a Γ-module, where N is the order of the group Γ. Since the quotient $\mathbb{C}[x,y]/I$ are isomorphic as Γ-modules for all I in a given connected component of $(\mathbb{C}^2)^{[nN],\Gamma}$, the variety $Y_{\Gamma,n}$ is non-singular, and is a union of components of $(\mathbb{C}^2)^{[nN],\Gamma}$. It turns out that $Y_{\Gamma,n}$ is connected as we will explain later.

Given $J \in \mathbb{C}^{2n}//\Gamma_n$, we regard it as an ideal in $\mathbb{C}[\mathbf{x},\mathbf{y}]$ of length $N^n n!$ (which is the order of Γ_n), where $\mathbf{x}$ and $\mathbf{y}$ stand for the coordinates $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ in $\mathbb{C}^{2n}$. Then the quotient $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ affords the regular representation of Γ_n, and its only Γ_n-invariants are constants. Thus for a given $f \in \mathbb{C}[x,y]^\Gamma$, we have $\sum_{i=1}^n f(x_i, y_i) = c_f \mod J$ for some constant c_f.

Denote by Γ_{n-1} the subgroup of Γ_n acting on $\mathbb{C}[x_2, \ldots, x_n, y_2, \ldots, y_n]$. We can show by induction that the algebra of invariants $\mathbb{C}[x_2, \ldots, x_n, y_2, \ldots, y_n]^{\Gamma_{n-1}}$ is

generated by the polynomials $\sum_{i=2}^n f(x_i, y_i)$, where f runs over an arbitrary linear basis $\mathcal{B}$ for the space of invariants $\mathbb{C}[x,y]^\Gamma$, (cf. [**Wa2**]). In the case when Γ is trivial, this is a theorem of Weyl. It follows that the space $\mathbb{C}[\mathbf{x},\mathbf{y}]^{\Gamma_{n-1}}$ is generated by x_1, y_1 and $\sum_{i=2}^n f(x_i, y_i)$, where $f \in \mathcal{B}$. The latter is equal to $c_f - f(x_1, y_1) \bmod J$. Thus $(\mathbb{C}[\mathbf{x},\mathbf{y}]/J)^{\Gamma_{n-1}}$ is generated by x_1, y_1 and $c_f - f(x_1, y_1)$, where $f \in \mathbb{C}[x,y]^\Gamma$. It follows that

$$\mathbb{C}[x_1, y_1]/(J \cap \mathbb{C}[x_1, y_1]) \cong (\mathbb{C}[\mathbf{x},\mathbf{y}]/J)^{\Gamma_{n-1}}. \tag{4.3}$$

This space has dimension $nN = |\Gamma_n|/|\Gamma_{n-1}|$ because $(\mathbb{C}[\mathbf{x},\mathbf{y}]/J)^{\Gamma_{n-1}}$ can be identified with the space of Γ_{n-1}-invariants in the regular representation of Γ_n. The first copy of Γ in the Cartesian product $\Gamma^n \subset \Gamma_n$ commutes with Γ_{n-1} above. It follows from (4.3) that the quotient $\mathbb{C}[x_1, y_1]/(J \cap \mathbb{C}[x_1, y_1])$ as a Γ-module is isomorphic to R^n. That is, $J \cap \mathbb{C}[x_1, y_1]$ lies in $Y_{\Gamma,n}$.

The map φ can be also understood as follows. Let $\mathcal{U}_{\Gamma,n}$ be the universal family over $\mathbb{C}^{2n}//\Gamma_n$ which is a subvariety of the Hilbert scheme $(\mathbb{C}^{2n})^{[n!N^n]}$:

$$\begin{array}{ccc} \mathcal{U}_{\Gamma,n} & \longrightarrow & \mathbb{C}^{2n} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^{2n}//\Gamma_n & \longrightarrow & \mathbb{C}^{2n}/\Gamma_n \end{array}$$

It has a natural Γ_n-action fiberwise such that each fiber carries the regular representation of Γ_n. Then $\mathcal{U}_{\Gamma,n}/\Gamma_{n-1}$ is flat and finite of degree nN over $\mathbb{C}^{2n}//\Gamma_n$, and thus can be identified with a family of subschemes of $\mathbb{C}^2$ as above. Then φ is the morphism given by the universal property of the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$ for the family $\mathcal{U}_{\Gamma,n}/\Gamma_{n-1}$.

It is observed [**Wa2**] that the variety of Γ-fixed-points $((\mathbb{C}^2)^{nN}/S_{nN})^\Gamma$ can be naturally identified with the wreath product orbifold $\mathbb{C}^{2n}/\Gamma_n$. Thus the Hilbert-Chow morphism $(\mathbb{C}^2)^{[nN]} \to (\mathbb{C}^2)^{nN}/S_{nN}$ induces a morphism between the varieties of Γ-fixed-points $\varsigma_n : Y_{\Gamma,n} \to \mathbb{C}^{2n}/\Gamma_n$. As a Γ-fixed-point set, $Y_{\Gamma,n}$ has an induced holomorphic symplectic 2-form from the Hilbert scheme $(\mathbb{C}^2)^{[nN]}$. It is easy to see that over the set of generic points in $\mathbb{C}^{2n}/\Gamma_n$ which consist of Γ_n-regular orbits on $\mathbb{C}^{2n}$, the morphism ς_n is one-to-one. It follows that ς_n is a crepant resolution of singularities.

Theorem 4.2. [**Wa2**] *There exists a natural morphism $\varphi : \mathbb{C}^{2n}//\Gamma_n \longrightarrow Y_{\Gamma,n}$ such that the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{C}^{2n}//\Gamma_n & \xrightarrow{\varphi} & Y_{\Gamma,n} \\ {\scriptstyle\sigma_n}\big\downarrow & & {\scriptstyle\varsigma_n}\big\downarrow \\ \mathbb{C}^{2n}/\Gamma_n & = & \mathbb{C}^{2n}/\Gamma_n \end{array}$$

Here $\varsigma_n : Y_{\Gamma,n} \to \mathbb{C}^{2n}/\Gamma_n$ is a crepant resolution of singularities. In particular φ is bijective over the set of generic points on $\mathbb{C}^{2n}//\Gamma_n$ consisting of the regular Γ_n orbits on $\mathbb{C}^{2n}$.

It is of great iterest to clarify the precise relations among $\mathbb{C}^{2n}//\Gamma_n, Y_{\Gamma,n}$ and $(\mathbb{C}^2//\Gamma)^{[n]}$. In an optimistic way, it was conjectured in [**Wa2**] that the morphism $\varphi : \mathbb{C}^{2n}//\Gamma_n \longrightarrow Y_{\Gamma,n}$ is an isomorphism over $\mathbb{C}^{2n}/\Gamma_n$. In the Appendix, we explain that $Y_{\Gamma,n}$ admits a quiver variety description in the sense of Nakajima [**Na1,**

Na3] (cf. [**Wa2**]). In particular this implies that $Y_{\Gamma,n}$ is connected. According to Nakajima (unpublished), the Hilbert scheme $(\mathbb{C}^2//\Gamma)^{[n]}$ also admits the same quiver variety description as $Y_{\Gamma,n}$ but with a different stability condition. It follows from Nakajima's results on quiver varieties (Corollary 4.2, [**Na1**]) that $Y_{\Gamma,n}$ is diffeomorphic to $(\mathbb{C}^2//\Gamma)^{[n]}$.

When $n = 1$, a theorem of Ito-Nakamura [**INr1**] says that φ is an isomorphism. When Γ is trivial, $\phi : \mathbb{C}^{2n}//S_n \to (\mathbb{C}^2)^{[n]}$ being an isomorphism is a very difficult theorem due to Haiman [**Hai2**]. Our construction above of the morphism φ was inspired by his construction. Indeed Haiman's theorem is equivalent to the validity of the Garsia-Haiman $n!$ conjecture, which in turn implies the Macdonald positivity conjecture [**Hai1, Mac**]. Our work provides a natural geometric framework which may allow a generalization (even for Γ cyclic) of these combinatorial results. This will be pursued elsewhere.

4.4. Equivalence of derived categories. This subsection may be regarded as a continuation of Subsect. 4.2. I thank A. King for stimulating discussions. In this subsection, we assume that Y is a (quasi-)projective surface acted by a finite group Γ with isolated singularities. Let $\tau : X \to Y/\Gamma$ be thecrepant resolution. Let $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ be the induced resolution of singularities.

Recall the Hilbert scheme of regular orbits $X^n//S_n$ was defined in the previous subsection. Denote by $\mathcal{Z}$ the universal closed subscheme $\mathcal{Z} \subset (X^n//S_n) \times X^n$ and denote by p, q the projections to X^n and $X^n//S_n$. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{q} & X^n \\ \downarrow{\scriptstyle p} & & \downarrow \\ X^n//S_n & \longrightarrow & X^n/S_n \end{array}$$

Haiman's theorem [**Hai2**] allows us to identify $X^n//S_n \cong X^{[n]}$, which we will use below implicitly. We denote by $D_{\Gamma_n}(Y^n)$ the bounded derived category of Γ_n-equivariant coherent sheaves on Y^n, and denote by $D(X^{[n]})$ the bounded derived category of coherent sheaves on $X^{[n]}$. Define two functors $\Phi : D(X^{[n]}) \to D_{S_n}(X^n)$ and $\Psi : D_{S_n}(X^n) \to D(X^{[n]})$ by

$$\begin{aligned} \Phi(-) &= Rp_*(\mathcal{O}_{\mathcal{Z}} \bigotimes q^*(-)) \\ \Psi(-) &= (Rq_* RHom(\mathcal{O}_{\mathcal{Z}}, p^*(-)))^{S_n}. \end{aligned}$$

By applying a remarkable result of Bridgeland, King and Reid [**BKR**] together with Haiman's isomorphism $X^n//S_n \cong X^{[n]}$, we see that Φ is an equivalence of categories and Ψ is its adjoint functor.

In a similar (and easier) way [**KV, BKR**] (also cf. [**GSV**]), we also have an equivalence of categories $\phi : D(X) \to D_\Gamma(Y)$. This induces an equivalence of categories $\phi^n : D(X^n) \to D_{\Gamma^n}(Y^n)$ by passing to the n-th Cartesian product. If we restrict ourselves to the S_n-equivariant sheaves, we obtain from the equivalence ϕ^n an equivalence of categories between $D_{S_n}(X^n)$ and $D_{\Gamma_n}(Y^n)$ (note that a sheaf on Y^n which is equivariant with respect to Γ^n and S_n is precisely a sheaf which is equivariant with respect to the wreath product $\Gamma_n = \Gamma^n \rtimes S_n$).

Combining with the equivalence Φ, we have established the following.

THEOREM 4.3. *Assume that Y is a (quasi-)projective surface acted by a finite group Γ with isolated singularities. Let $\tau : X \to Y/\Gamma$ be the crepant resolution. There exists an equivalence of categories between $D_{\Gamma_n}(Y^n)$ and $D(X^{[n]})$ associated to the resolution of singularities $\tau_n : X^{[n]} \to Y^n/\Gamma_n$.*

In other words, we can say that if $\tau : X \to Y/\Gamma$ is good in the sense of equivalence of derived categories then $\tau_n : X^{[n]} \to Y^n/\Gamma_n$ is also good. This settles some questions and confusing points in [**Wa2**]. In the case when Γ is trivial and X equals Y, it reduces to the equivalence between $D(X^{[n]})$ and $D_{S_n}(X^n)$.

4.5. Cup product on $X^{[n]}$ vs convolution product on Γ_n. So far we have seen there is a surprising analog (cf. Eqns. (2.2) and (3.3)) between the convolution product $K_i(c,n)$ (at least when $i = 0, 1$) on the representation ring $R(\Gamma_n)$ and the cup product with $B_i(\gamma, n)$ in the cohomology ring $H^*(X^{[n]})$. In the case when Γ is a finite subgroup of $SL_2(\mathbb{C})$, and X is the minimal resolution $\mathbb{C}^2//\Gamma$ of the simple singularity $\mathbb{C}^2/\Gamma$, we have expected a precise connection between these two products which will generalize the isomorphism of vector spaces between $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ and $R(\Gamma_n)$.

In the case when Γ is trivial, Lehn and Sorger [**LS**] indeed found a precise and canonical connection between them. This result has been also obtained by Vasserot [**Vas**] by a different technique. Let us associate a degree $d = n - \ell(\lambda)$ to a permutation σ of cycle type λ. This defines a grading on $\mathbb{C}[S_n]$. The convolution product does not preserve this grading but preserves the filtration on $R(S_n)$

$$F^0\mathbb{C}[S_n] \subset F^1\mathbb{C}[S_n] \subset \cdots \subset F^{n-1}\mathbb{C}[S_n] = \mathbb{C}[S_n],$$

where $F^d\mathbb{C}[S_n]$ is the space spanned by permutations of degree $\leq d$. We call the induced product on $\mathbb{C}[S_n] = \oplus_{d=0}^{n-1} F^d\mathbb{C}[S_n]/F^{d-1}\mathbb{C}[S_n]$ the *filtered convolution product*, and denote it by $\cup$. In other words, given two permutations σ and π, we see that $\sigma \cup \pi$ coincides with the usual convolution product $\sigma * \pi$ when $\deg(\sigma) + \deg(\pi) = \deg(\sigma\pi)$ and is 0 otherwise.

We may restrict the filtered convolution to $R(S_n) \subset \mathbb{C}[S_n]$. Recall a linear basis of $H^*((\mathbb{C}^2)^{[n]})$ is given by $\mathfrak{q}_\lambda = \prod_{r \geq 1} \mathfrak{q}_r^{m_r}|0\rangle$ while a linear basis of $R(S_n)$ is given by $p_\lambda = \prod_{r \geq 1} p_r^{m_r}|0\rangle$, where $\lambda = (1^{m_1}2^{m_2}\ldots)$ runs over all partitions of n. Here we also use $|0\rangle$ to denote $1 \in R(\Gamma_0) \cong \mathbb{C}$.

THEOREM 4.4. [**LS, Vas**] *The linear map from cohomology ring $H^*((\mathbb{C}^2)^{[n]})$ to the filtered representation ring $R(S_n)$ by sending $\mathfrak{q}_\lambda$ to p_λ is a graded ring isomorphism.*

Note that the formula (2.2) holds for a projective surface X. For the affine plane which is quasi-projective, certain degeneracy occurs and the modified formula becomes [**Lehn**]

$$\mathfrak{d} = -\frac{1}{2}\sum_{n,m>0} nm\mathfrak{q}_{n+m}\frac{\partial}{\partial \mathfrak{q}_n}\frac{\partial}{\partial \mathfrak{q}_m}. \tag{4.4}$$

Let us briefly comment on how Lehn and Sorger proved their theorem. If one examines the proof of the equation (3.3), one sees that the operator acting on $R(S_n)$ which corresponds to $\mathfrak{d}$ given in (4.4) is the filtered convolution product with the conjugacy class of cycle type $(1^{n-2}2)$. With the help of the two parallel formulas (2.3) and (3.1), one can establish that the cup product with $c(p_{1*}\mathcal{O}_{\mathcal{Z}_n})$

in $H^*((\mathbb{C}^2)^{[n]})$ corresponds exactly to the filtered convolution product with ε_n in $R(S_n)$. These two statements are mainly responsible for establishing the theorem above. Indeed, Lehn-Sorger managed to go further to establish a stronger form of the above theorem which claims the graded ring isomorphism is valid between $H^*((\mathbb{C}^2)^{[n]}, \mathbb{Z})$ and $R_{\mathbb{Z}}(S_n)$.

In [**CR**], Chen and Ruan introduced the orbifold cohomology ring for a general orbifold which is not necessarily a global quotient by a finite group. Ruan observed that the filtered convolution product on $R(S_n)$ above coincides with the orbifold cup product on the orbifold cohomology ring of the orbifold $\mathbb{C}^{2n}/S_n$, cf. [**Ruan**]. Therefore by combining with the theorem above, the cohomology ring on the Hilbert scheme $(\mathbb{C}^2)^{[n]}$ is isomorphic to the orbifold cohomology ring of the symmetric product $\mathbb{C}^{2n}/S_n$. This raises the question how to relate the cohomology ring on the Hilbert schemes to the orbifold cohomology ring on the symmetric product or more generally on the wreath product orbifolds.

5. Appendix: A quiver variety description of $Y_{\Gamma,n}$

We recall (cf. [**Na3**]) that the Hilbert scheme $(\mathbb{C}^2)^{[K]}$ of K points in $\mathbb{C}^2$ admits a description in terms of a quiver consisting of one vertex and one arrow starting and ending at that vertex. More explicitly, we denote

$$\widetilde{H}(K) = \left\{ \begin{array}{c|l} & i)[B_1, B_2] + ij = 0 \\ (B_1, B_2, i, j) & ii) \text{ there exists no proper subspace} \\ & S \subset \mathbb{C}^K \text{ such that } B_\alpha(S) \subset S \text{ and} \\ & \text{im } i \subset S \quad (\alpha = 1, 2) \end{array} \right\},$$

where $B_1, B_2 \in \mathrm{End}(\mathbb{C}^K), i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^K), j \in \mathrm{Hom}(\mathbb{C}^K, \mathbb{C})$. Then we have an isomorphism

$$(\mathbb{C}^2)^{[K]} \cong \widetilde{H}(K)/GL_K(\mathbb{C}), \tag{5.1}$$

where the action of $GL_K(\mathbb{C})$ on $\widetilde{H}(K)$ is given by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

It is also often convenient to regard (B_1, B_2) to be in $\mathrm{Hom}(\mathbb{C}^K, \mathbb{C}^2 \otimes \mathbb{C}^K)$. We remark that one may drop j in the above formulation because one can show by using the stability condition that $j = 0$ (cf. [**Na3**], Proposition 2.8).

The bijection in (5.1) is given as follows. For $I \in (\mathbb{C}^2)^{[K]}$, i.e. an ideal in $\mathbb{C}[x, y]$ of colength K, the multiplication by x, y induces endomorphisms B_1, B_2 on the K-dimensional quotient $\mathbb{C}[x, y]/I$, and the homomorphism $i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^K)$ is given by letting $i(1) = 1 \bmod I$. Conversely, given (B_1, B_2, i), we define a homomorphism $\mathbb{C}[x, y] \to \mathbb{C}^K$ by $f \mapsto f(B_1, B_2)i(1)$. The stability condition is equivalent to the homomorphism $\mathbb{C}[x, y] \to \mathbb{C}^K$ being surjective, which implies that the kernel I of this homomorphism is an ideal of $\mathbb{C}[x, y]$ of length K. One easily checks that the two maps are inverse to each other.

Set $K = nN$, where N is the order of Γ. We may identify $\mathbb{C}^K$ with the direct sum of n copies of the regular representation R of Γ, $\mathbb{C}^2$ with the defining representation Q of Γ by the embedding $\Gamma \subset SL_2(\mathbb{C})$, and $\mathbb{C}$ with the trivial representation V_{γ_0} of Γ. Denote by

$$M(n) = \mathrm{Hom}(R^n, Q \otimes R^n) \bigoplus \mathrm{Hom}(V_{\gamma_0}, R^n) \bigoplus \mathrm{Hom}(R^n, V_{\gamma_0}).$$

By definition $\widetilde{H}(nN) \subset M(n)$. Let $GL_\Gamma(R)$ be the group of Γ-equivariant automorphisms of R. Then the group $G \equiv GL_\Gamma(R^n)$ acts on the Γ-invariant subspace $M(n)^\Gamma$. We give the following description of $Y_{\Gamma,n}$ as a quiver variety. This result is certainly known to Nakajima and it seems to be observed by others as well. A proof of it is given in [**Wa2**].

THEOREM 5.1. *The variety $Y_{\Gamma,n}$ admits the following description:*

$$Y_{\Gamma,n} \cong (\widetilde{H}(nN) \cap M(n)^\Gamma)/GL_\Gamma(R^n).$$

Consider the Γ-module decomposition $Q \otimes V_{\gamma_i} = \bigoplus_j a_{ij} V_{\gamma_j}$, where $a_{ij} \in \mathbb{Z}_+$, and V_{γ_i} $(i = 0, \ldots, r)$ are irreducible representations corresponding to the characters γ_i of Γ. Set $\dim V_{\gamma_i} = n_i$. Then

$$\begin{aligned}
(5.2)\ M(n)^\Gamma &= \mathrm{Hom}_\Gamma(R^n, Q \otimes R^n) \bigoplus \mathrm{Hom}_\Gamma(V_{\gamma_0}, R^n) \bigoplus \mathrm{Hom}_\Gamma(R^n, V_{\gamma_0}) \\
&\cong \mathrm{Hom}_\Gamma(\sum_i \mathbb{C}^{nn_i} \otimes V_{\gamma_i}, \mathbb{C}^2 \otimes \sum_i \mathbb{C}^{nn_i} \otimes V_{\gamma_i}) \\
&\quad \bigoplus \mathrm{Hom}_\Gamma(V_{\gamma_0}, R^n) \bigoplus \mathrm{Hom}_\Gamma(R^n, V_{\gamma_0}) \\
&\cong \sum_{ij} a_{ij} \mathrm{Hom}(\mathbb{C}^{nn_i}, \mathbb{C}^{nn_j}) \bigoplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}^n) \bigoplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}).
\end{aligned}$$

where Hom_Γ stands for the Γ-equivariant homomorphisms. In the language of quiver varieties as formulated by Nakajima [**Na1, Na4**], the above description of $Y_{\Gamma,n}$ identifies $Y_{\Gamma,n}$ with a quiver variety associated to the following data: the graph consists of the same vertices and edges as the McKay quiver which is an affine Dynkin diagram associated to a finite subgroup Γ of $SL_2(\mathbb{C})$; the vector space V_i associated to the vertex i is isomorphic to the direct sum of n copies of the i-th irreducible representation V_{γ_i}; the vector space $W_i = 0$ for nonzero i and $W_0 = \mathbb{C}$.

The variety $Y_{\Gamma,n}$ is connected since a quiver variety is always connected [**Na1, Na4**]. According to Nakajima (unpublished), the Hilbert scheme $(\mathbb{C}^2//\Gamma)^{[n]}$ admits a quiver variety description in terms of the same quiver data as specified in the above paragraph but with a different stability condition. It follows by Corollary 4.2 of [**Na1**] that $(\mathbb{C}^2//\Gamma)^{[n]}$ and $Y_{\Gamma,n}$ is diffeomorphic.

Table 1. A DICTIONARY

Hilbert Scheme $X^{[n]}$ Picture	Wreath Product Γ_n Picture
length-n schemes supported at a point	n-cycle
cohomology group $H^*(X^{[n]})$	Grothendieck group $R(\Gamma_n)$
$\mathbb{H}_X \cong$ Heisenberg Fock space	$\mathbb{R}_\Gamma \cong$ Heisenberg Fock space
lattice $H^*(X, \mathbb{Z})/tor$	lattice $R_{\mathbb{Z}}(\Gamma)$
$(\mathbb{C}^2)^{[n]}$	S_n
$(\mathbb{C}^2//\Gamma)^{[n]}$ for $\Gamma \subset SL_2(\mathbb{C})$	Γ_n for $\Gamma \subset SL_2(\mathbb{C})$
cup product (X closed)	convolution product
cup product (X non-closed)	(filtered) convolution product
correspondence varieties	induction/restriction functors etc
boundary of $X^{[n]}$	conjugacy class of type $(1^{n-2}2)$
cohomology class $B_i(\alpha, n)$	conjugacy class $K_i(c, n)$
total Chern class $c(L^{[n]})$	character $\varepsilon_n(\gamma)$
?	character $\eta_n(\gamma)$
cohomology class $G_i(\alpha, n)$	?

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